

Λ -symmetries of Dynamical Systems, Hamiltonian and Lagrangian equations

Giampaolo Cicogna

Dipartimento di Fisica, Università di Pisa
and *INFN, Sezione di Pisa,*
Largo B. Pontecorvo 3, 50127 Pisa (Italy)
`cicogna@df.unipi.it`

Abstract

After a brief survey of the definition and the properties of Λ -symmetries in the general context of dynamical systems, the notion of “ Λ -constant of motion” for Hamiltonian equations is introduced. If the Hamiltonian problem is derived from a Λ -invariant Lagrangian, it is shown how the Lagrangian Λ -invariance can be transferred into the Hamiltonian context and shown that the Hamiltonian equations turn out to be Λ -symmetric. Finally, the “partial” (Lagrangian) reduction of the Euler-Lagrange equations is compared with the reduction obtained for the corresponding Hamiltonian equations.

1 Introduction (λ -symmetries)

Let me briefly recall for the reader’s convenience the basic definition of λ -symmetry (with lower case λ), originally introduced by C. Muriel and J.L. Romero in 2001 [1, 2].

Let me consider the simplest case of a single ODE $\Delta(t, u(t), \dot{u}, \ddot{u} \dots) = 0$ for the unknown function $u = u(t)$ (I will denote by t the independent variable, with the only exception of the final Section 4, because the applications I am going to propose will concern the case of Dynamical Systems (DS), where the independent variable is precisely the time t , and $\dot{u} = du/dt$, etc.). Given a vector field

$$X = \varphi(u, t) \frac{\partial}{\partial u} + \tau(u, t) \frac{\partial}{\partial t}$$

the idea is to suitably *modify* its prolongation rules. The first λ -prolongation $X_\lambda^{(1)}$ is the defined by

$$X_\lambda^{(1)} = X^{(1)} + \lambda(\varphi - \tau\dot{u}) \frac{\partial}{\partial \dot{u}} \quad (1)$$

where $\lambda = \lambda(u, \dot{u}, t)$ is a C^∞ function, and $X^{(1)}$ is the standard first prolongation. Other modifications have to be introduced for higher prolongations, but in the present paper I need only just the first one.

An n -th order ODE $\Delta = 0$ is said to be λ -invariant under $X_\lambda^{(n)}$ if

$$X_\lambda^{(n)} \Delta \Big|_{\Delta=0} = 0$$

where $X_\lambda^{(n)}$ is the n -th λ -prolongation of X .

It should be emphasized that λ -symmetries are not properly symmetries, because they do not transform in general solutions of a λ -invariant equation into solutions, nevertheless they share with standard Lie point-symmetries some important properties, namely: if an equation is λ -invariant, then

- the order of the equation can be lowered by one
- invariant solutions can be found (notice that conditional symmetries do the same, but λ -symmetries are clearly *not* conditional symmetries)
- convenient new (“symmetry adapted”) variables can be suggested.

In the context of DS, which is the main object of this paper, the first two properties are not effective, the third one is instead one of my starting points.

Before considering the role of λ -symmetries in DS, let me recall that many applications and extensions of this notion have been proposed in these 10 years: these include extensions to systems of ODE’s, to PDE’s, applications to variational principles and Noether-type theorems, the analysis of their connections with nonlocal symmetries, with symmetries of exponential type, with hidden, or “lost” symmetries, with potential, telescopic symmetries as well. Other investigations concern their deep geometrical interpretation, with the introduction of a suitable notion of deformed Lie differential operators, the study of their dynamical effects in terms of changes of reference frames, and so on. Only the papers more directly involved with the argument considered in this paper will be quoted; for a fairly complete list of references see e.g. [3, 4, 5]. A very recent application concerns discrete difference equations [6].

2 Λ -symmetries for DS

I am going to consider the case of dynamical systems, i.e. systems of first-order ODE’s

$$\dot{u}_a = f_a(u, t) \quad (a = 1, \dots, m)$$

for the $m > 1$ unknowns $u_a = u_a(t)$.

Let me start with a trivial (but significant) case: if the DS admits a rotation symmetry, then it is completely natural to introduce as new variables the radius r and the angle θ , and the DS immediately takes a simplified form, as well known. However, in general, symmetries of DS may be very singular, and/or difficult to detect. An example can be useful: the DS

$$\dot{u}_1 = u_1 u_2 \quad \dot{u}_2 = -u_1^2$$

admits the (not very useful or illuminating) symmetry generated by (with $r^2 = u_1^2 + u_2^2$)

$$X = \left(\frac{2u_1}{r^2} - \frac{u_1 u_2}{r^3} \log \frac{u_2 - r}{u_2 + r} \right) \frac{\partial}{\partial u_1} + \left(\frac{2u_2}{r^2} - \frac{u_1^2}{r^3} \log \frac{u_2 - r}{u_2 + r} \right) \frac{\partial}{\partial u_2} .$$

In this example the rotation (with a commonly accepted abuse of language, the same symbol X denotes both the symmetry and its Lie generator)

$$X = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2}$$

is a λ -symmetry (its precise definition will be given in a moment), and *not* a symmetry in the “standard” sense; *nevertheless*, still introducing the variables as before, i.e. r and θ , the DS takes the very simple form

$$\dot{r} = 0 \quad \dot{\theta} = -r \cos \theta .$$

This is just a first, simple example of the possible role of λ -symmetries in the context of DS.

2.1 Λ -symmetries of general DS

The natural way to extend the definition (1) of the first λ -prolongation of the vector field

$$X = \varphi_a(u, t) \frac{\partial}{\partial u_a} + \tau(u, t) \frac{\partial}{\partial t} = \varphi \cdot \nabla_u + \tau \partial_t$$

to the case of $m > 1$ variables u_a is the following (sum over repeated indices)

$$X_\Lambda^{(1)} = X^{(1)} + \Lambda_{ab}(\varphi_b - \tau \dot{u}_b) \cdot \nabla_{\dot{u}_a}$$

where now $\Lambda = \Lambda(t, u_a, \dot{u}_a)$ is a $(m \times m)$ matrix; accordingly, I denote by the upper case Λ these symmetries in this context.

To simplify, let me assume from now on $\tau = 0$ (or use evolutionary vector field, it is not restrictive).

Then the given DS is Λ -invariant under X (or X is a Λ -symmetry for the DS), i.e. $X_\Lambda^{(1)}(\dot{u} - f)|_{\dot{u}=f} = 0$, if and only if

$$[f, \varphi]_a + \partial_t \varphi_a = -(\Lambda \varphi)_a \quad (a = 1, \dots, m)$$

where

$$[f, \varphi]_a \equiv f_b \nabla_{u_b} \varphi_a - \varphi_b \nabla_{u_b} f_a .$$

Given X , we now introduce the following new $m+1$ “canonical” (or *symmetry-adapted*) variables (*notice that they are independent of Λ*): precisely, $m-1$ variables $w_j = w_j(u)$ which, together with the time t , are X -invariant:

$$X w_j = X t = 0 \quad (j = 1, \dots, m-1)$$

and the coordinate z , “rectifying” the action of X , i.e.

$$X = \frac{\partial}{\partial z} .$$

Writing the given DS in these new variables, we obtain a “reduced” form of the DS, as stated by the following theorem [7, 8, 9].

Theorem 1 *Let X be a Λ -symmetry for a given DS; once the DS is written in terms of the new variables w_j, z, t , i.e.*

$$\dot{w}_j = W_j(w, z, t) \quad \dot{z} = Z(w, z, t)$$

the dependence on z of the r.h.s. W_j, Z is controlled by the formulas

$$\frac{\partial W_j}{\partial z} = \frac{\partial w_j}{\partial u_a} (\Lambda\varphi)_a \equiv M_j \quad \frac{\partial Z}{\partial z} = \frac{\partial z}{\partial u_a} (\Lambda\varphi)_a \equiv M_m .$$

One has:

- *If $\Lambda = 0$ then $M_j = M_m = 0$ and W_j, Z are independent of z*
- *If $\Lambda = \lambda I$ then only Z depends on z*
- *Otherwise, a “partial” reduction is obtained: If some $M_k = 0$, then W_k is independent of z . In terms of the new variables, the Λ -prolongation becomes*

$$X_\Lambda^{(1)} = \frac{\partial}{\partial z} + M_j \frac{\partial}{\partial \dot{w}_j} + M_m \frac{\partial}{\partial \dot{z}} .$$

The first case ($\Lambda = 0$) clearly means that X is an *exact*, or standard Lie point-symmetry [7]; the second one has been considered in detail by Muriel and Romero [8] (notice that actually it would be enough to require $\Lambda\varphi = \lambda\varphi$); the last case has been dealt with in [9]: several situations can be met, depending on the number of vanishing M_j (e.g., one may obtain triangular DS, or similar).

2.2 Hamiltonian DS

I now consider the special case in which the DS is a *Hamiltonian* DS. Obvious changes in the notations can be introduced: the m variables $u = u_a(t)$ are replaced by the $m = 2n$ variables $q_\alpha(t), p_\alpha(t)$ ($\alpha = 1, \dots, n$), and the DS is now the system of the Hamiltonian equations of motions for the given Hamiltonian $H = H(q, p, t)$:

$$\dot{u} = J\nabla H \equiv F(u, t) \quad , \quad \nabla \equiv \nabla_u \equiv (\nabla_q, \nabla_p)$$

where J is the standard symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} .$$

A vector field X can be written accordingly (with $a = 1, \dots, 2n$; $\alpha = 1, \dots, n$)

$$X = \varphi_\alpha(u, t) \frac{\partial}{\partial q_\alpha} + \psi_\alpha(u, t) \frac{\partial}{\partial p_\alpha} \equiv \Phi \cdot \nabla_u \quad , \quad \Phi \equiv (\varphi_\alpha, \psi_\alpha)$$

and all the above discussion clearly holds if X is a Λ -symmetry for an Hamiltonian DS. Clearly, here Λ is a $(2n \times 2n)$ matrix. But Hamiltonian problems possess certainly a *richer* structure with respect to general DS, which deserves to be exploited; a first instance is clearly provided by the notion of conservation rules, with its related topics.

Let me then distinguish two cases:

(i) X admits a *generating function* $G(u, t)$ (then X is often called a “Hamiltonian symmetry”):

$$\Phi = J \nabla G \quad \text{i.e.} \quad \varphi = \nabla_p G, \quad \psi = -\nabla_q G \quad (2)$$

this implies $\nabla D_t G = 0$, where D_t is the total derivative, i.e. G is a constant of motion, $D_t G = 0$, possibly apart from an additional time-dependent term, as well known.

(ii) X does not admit a generating function: also in this case, defining

$$S(u, t) \equiv \nabla \cdot \Phi \quad \text{one has that} \quad D_t S = 0 \quad (3)$$

and therefore, if $S \neq \text{const}$, then S is a first integral (the examples known to me of first integrals of this form are rather tricky, being usually obtained multiplying symmetries by first integrals; but they “in principle” exist, and their presence will be important for the following discussion, see subsect. 3.4).

Direct calculations can show the following:

Theorem 2 *If the Hamiltonian equations of motion admit a Λ -symmetry X with a matrix Λ , then:*

in case (i)

$$\nabla(D_t G) = J \Lambda \Phi = J \Lambda J \nabla G$$

in case (ii)

$$D_t S = -\nabla(\Lambda \Phi) .$$

When this happens, G (resp. S) will be called a “ Λ -constant of motion”.

If $\Lambda = 0$, i.e. when X is a “standard” (or “exact”) symmetry, the above equations become clearly the usual conservation rules; Λ -symmetries can then be viewed as “perturbations” of the exact symmetries. More explicitly, the equations in Theorem 2 state the precise “deviation” from the conservation of G (resp. of S) due to the fact that the invariance under X is “broken” by the presence of a nonzero matrix Λ .

As a special case for case (i), the following Corollary may be of interest:

Corollary 1 *Under mild assumptions ($\Lambda \Phi = \lambda \Phi$, $\lambda = \lambda(G)$), then the Λ -constant of motion G satisfies a “completely separated equation”, involving only $G(t)$:*

$$\dot{G} = \gamma(t, G) .$$

This equation expresses how much the conservation of $G(t)$ is “violated” along the time evolution. If Λ is in some sense “small”, then G is “almost” conserved.

3 When a Λ -symmetry of the Hamiltonian equations is inherited by a Λ -invariant Lagrangian

3.1 Λ -invariant Lagrangians, Noether theorem and Λ -conservation rules

Let me consider (for simplicity) only first-order Lagrangians:

$$\mathcal{L} = \mathcal{L}(q_\alpha, \dot{q}_\alpha, t) \quad (\alpha = 1, \dots, n)$$

Such a Lagrangian is said to be $\Lambda^{(\mathcal{L})}$ -invariant[10, 11] under

$$X^{(\mathcal{L})} = \varphi_\alpha(q, t) \frac{\partial}{\partial q_\alpha} = \varphi \cdot \nabla_q$$

if there is an $(n \times n)$ matrix

$$\Lambda^{(\mathcal{L})} = \Lambda^{(\mathcal{L})}(q, \dot{q}, t)$$

such that

$$\left(X_\Lambda^{(\mathcal{L})} \right)^{(1)}(\mathcal{L}) = 0$$

where $\left(X_\Lambda^{(\mathcal{L})} \right)^{(1)}$ is the first $\Lambda^{(\mathcal{L})}$ -prolongation of $X^{(\mathcal{L})}$ (the notation is rather heavy, to carefully distinguish the Lagrangian case from the Hamiltonian one, to be considered in the next subsection).

We then have [11]

Theorem 3 *If the Lagrangian \mathcal{L} is $\Lambda^{(\mathcal{L})}$ -invariant under $X^{(\mathcal{L})}$ then, putting*

$$\mathcal{P}_{\alpha\beta} = \varphi_\alpha p_\beta \quad \text{with} \quad p_\beta = \frac{\partial \mathcal{L}}{\partial \dot{q}_\beta}$$

one has

$$D_t \mathbf{P} = -\Lambda_{\alpha\beta}^{(\mathcal{L})} \varphi_\beta \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} = -(\Lambda^{(\mathcal{L})} \varphi)_\alpha p_\alpha$$

where $\mathbf{P} = \text{Tr}(\mathcal{P}) = \varphi_\alpha p_\alpha$; or also, introducing a “deformed derivative” \widehat{D}_t

$$(\widehat{D}_t)_{\alpha\beta} \equiv D_t \delta_{\alpha\beta} + \Lambda_{\alpha\beta}^{(\mathcal{L})} \quad \text{then} \quad \text{Tr}(\widehat{D}_t \mathcal{P}) = 0 .$$

This result can be called the “*Noether $\Lambda^{(\mathcal{L})}$ -conservation rule*”. Indeed, if $\Lambda^{(\mathcal{L})} = 0$, the standard Noether theorem is recovered.

In the special case $\Lambda^{(\mathcal{L})}\varphi = \lambda\varphi$, the above result becomes

$$\widehat{D}_t \mathbf{P} = 0 \quad \text{where} \quad \widehat{D}_t = D_t + \lambda .$$

Theorem 3 can be extended [11] to divergence symmetries and to generalized symmetries as well. Also, higher-order Lagrangians can be included: the $\Lambda^{(\mathcal{L})}$ -conservation rule has the same form, but $\mathcal{P}_{\alpha\beta}$ is different: for instance, for second-order Lagrangians one has

$$\mathcal{P}_{\alpha\beta} = \varphi_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\beta} + ((\widehat{D}_t)_{\alpha\gamma} \varphi_\gamma) \frac{\partial \mathcal{L}}{\partial \ddot{q}_\beta} - \varphi_\alpha D_t \frac{\partial \mathcal{L}}{\partial \ddot{q}_\beta} .$$

3.2 From Lagrangians to Hamiltonians

Assume to have a Lagrangian which is $\Lambda^{(\mathcal{L})}$ -invariant under a vector field

$$X^{(\mathcal{L})} = \varphi_\alpha \frac{\partial}{\partial q_\alpha}$$

and introduce the corresponding Hamiltonian H with its Hamiltonian equations of motion. The natural question is whether the $\Lambda^{(\mathcal{L})}$ -symmetry $X^{(\mathcal{L})}$ of the Lagrangian is transferred to some $\Lambda^{(H)}$ -symmetry $X^{(H)}$ of the Hamiltonian equations of motion. Two problems then arise: *i*) to extend the vector field $X^{(\mathcal{L})}$ to a suitable vector field $X^{(H)}$, and *ii*) to extend the $(n \times n)$ matrix $\Lambda^{(\mathcal{L})}$ to a suitable $(2n \times 2n)$ matrix $\Lambda^{(H)}$.

First, the vector field $X^{(H)}$ is expected to have the form

$$X \equiv X^{(H)} = \varphi_\alpha \frac{\partial}{\partial q_\alpha} + \psi_\alpha \frac{\partial}{\partial p_\alpha} \quad (4)$$

where the coefficient functions ψ must be determined. This can be done observing that the variables p are related to \dot{q} (and then the first $\Lambda^{(\mathcal{L})}$ -prolongation of $X^{(\mathcal{L})}$ is needed, where the “effect” of $\Lambda^{(\mathcal{L})}$ is present). One finds, after some explicit calculations,

$$\psi_\alpha = \frac{\partial}{\partial \dot{q}_\alpha} \left(D_t \mathbf{P} + \Lambda_{\beta\gamma}^{(\mathcal{L})} \varphi_\gamma \frac{\partial \mathcal{L}}{\partial \dot{q}_\beta} \right) - \frac{\partial \Lambda_{\beta\gamma}^{(\mathcal{L})}}{\partial \dot{q}_\alpha} \varphi_\gamma \frac{\partial \mathcal{L}}{\partial \dot{q}_\beta} - p_\beta \frac{\partial \varphi_\beta}{\partial q_\alpha} . \quad (5)$$

But the term in parenthesis vanishes if the Lagrangian is $\Lambda^{(\mathcal{L})}$ -invariant, thanks to Theorem 3; in addition, if $\Lambda^{(\mathcal{L})}$ does not depend on \dot{q} (as happens in most cases, otherwise a separate treatment is needed, see subsect. 3.4), then we are left with

$$\psi_\alpha = -p_\beta \frac{\partial \varphi_\beta}{\partial q_\alpha} . \quad (6)$$

This implies that X admits a generating function, which is just

$$G = \varphi_\alpha p_\alpha \equiv \mathbf{P}$$

using the notations introduced in Theorem 3.

Second, let me now introduce the following $(2n \times 2n)$ matrix

$$\Lambda \equiv \Lambda^{(H)} = \begin{pmatrix} \Lambda^{(\mathcal{L})} & 0 \\ -\frac{\partial \Lambda^{(\mathcal{L})}}{\partial q_\alpha} p_\gamma & \Lambda^{(2)} \end{pmatrix}$$

where $\Lambda^{(2)}$ must satisfy (Λ is not uniquely defined, as well known)

$$\Lambda_{\alpha\beta}^{(2)} \frac{\partial \varphi_\gamma}{\partial q_\beta} = \Lambda_{\gamma\beta}^{(\mathcal{L})} \frac{\partial \varphi_\beta}{\partial q_\alpha}.$$

It is well known that Euler-Lagrange equations coming from a $\Lambda^{(\mathcal{L})}$ -invariant Lagrangian do *not exhibit in general* Λ -symmetry. In contrast with this, it is not difficult to verify explicitly that the Hamiltonian equations of motion turn out to be $\Lambda^{(H)}$ -symmetric under the vector field $X^{(H)}$ obtained according to the above prescriptions.

In conclusion, I have shown the following

Theorem 4 *If \mathcal{L} is a Λ -invariant Lagrangian under a vector field $X^{(\mathcal{L})}$ with a matrix $\Lambda^{(\mathcal{L})}$ (not depending on \dot{q}), one can extend $X^{(\mathcal{L})}$ to a vector field $X \equiv X^{(H)}$ and the $(n \times n)$ matrix $\Lambda^{(\mathcal{L})}$ to a $(2n \times 2n)$ matrix $\Lambda \equiv \Lambda^{(H)}$ in such a way that the resulting Hamiltonian equations of motion are Λ -symmetric under X ; in addition, $G = \varphi_\alpha p_\alpha$ is a Λ -constant of motion.*

Example 1 *The Lagrangian (with $n = 2$)*

$$\mathcal{L} = \frac{1}{2} \left(\frac{\dot{q}_1}{q_1} - q_1 \right)^2 + \frac{1}{2} (\dot{q}_1 - q_1 \dot{q}_2)^2 \exp(-2q_2) + q_1 \exp(-q_2)$$

is $\Lambda^{(\mathcal{L})}$ -invariant under

$$X^{(\mathcal{L})} = q_1 \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}$$

with

$$\Lambda^{(\mathcal{L})} = \text{diag}(q_1, q_1).$$

It is easy to write the Hamiltonian equations of motion and to check that they are indeed Λ -symmetric under

$$X = q_1 \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} - p_1 \frac{\partial}{\partial p_1}$$

with

$$\Lambda = \Lambda^{(H)} = \begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 \\ -p_1 & -p_2 & q_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

X -invariant coordinates are $w_1 = q_1 \exp(-q_2)$, $w_2 = q_1 p_1$, $w_3 = p_2$, and, as expected, the generating function $G = w_2 + w_3$ satisfies the Λ -conservation rule

$$\nabla_u D_t G = J \Lambda \Phi \quad \text{or} \quad D_t G = -q_1 G.$$

A special, but rather common, case is described by the following:

Corollary 2 *If*

$$\Lambda^{(\mathcal{L})}\varphi = c\varphi$$

where c is a constant, then also $\Lambda\Phi = c\Phi$ and the “most complete” reduction of the Hamiltonian equations of motion is obtained:

$$\dot{G} = \gamma(G, t) \quad \dot{w}_j = W_j(w, G, t) \quad \dot{z} = Z(w, G, z, t)$$

3.3 Reduction of the Euler-Lagrange equations versus the Hamiltonian equations

In this section I want to compare the reduction procedure which is provided by the presence of a Λ -symmetry of a Lagrangian (i.e. the reduction of Euler-Lagrange equations) with the analogous reduction of the Hamiltonian equations of motion.

Let me start recalling that any vector field $X = \varphi_\alpha \partial / \partial q_\alpha$ admits n (0-order) invariants (as already said, see subsect. 2.1)

$$w_j = w_j(q, t) \quad (j = 1, \dots, n-1) \quad \text{and the time } t$$

and n first-order differential invariants $\eta_\alpha = \eta_\alpha(q, t, \dot{q})$ under the first prolongation $X^{(1)}$

$$X^{(1)}\eta_\alpha = 0 \quad (\alpha = 1, \dots, n) .$$

Both if $X^{(1)}$ is standard and if it is a Λ prolongation (under the condition $\Lambda\varphi = \lambda\varphi$), it is well known that \dot{w}_j are $n-1$ first-order differential invariants (notice that this is an “algebraic” property, not related to dynamics). If one now chooses another independent first-order differential invariant $\zeta = \zeta(q, t, \dot{q})$, then one has that any first-order $\Lambda^{(\mathcal{L})}$ -invariant Lagrangian is a function of the above $2n$ invariants

$$t, w_j, \dot{w}_j \quad \text{and} \quad \zeta .$$

Writing the Lagrangian in terms of these variables, the Euler-Lagrange equation for ζ is then simply

$$\frac{\partial \mathcal{L}}{\partial \zeta} = 0 .$$

This first-order equation provides in general a “partial” reduction, i.e., it produces only *particular solutions*, even considering the Euler-Lagrange equations for the other variables [10, 3] (notice that this is true both for exactly invariant and for $\Lambda^{(\mathcal{L})}$ -invariant Lagrangians).

I want to emphasize that, introducing Λ -symmetric Hamiltonian equations of motion along the lines stated in Theorem 4, then a “better” reduction is obtained, and no solution is lost. The following example clarifies this point.

Example 2 *The Lagrangian ($n = 2$)*

$$\mathcal{L} = \frac{1}{2} \left(\frac{\dot{q}_1}{q_1} - \log q_1 \right)^2 + \frac{1}{2} \left(\frac{\dot{q}_1}{q_1} + \frac{\dot{q}_2}{q_2} \right)^2 \quad (q_1 > 0)$$

is $\Lambda^{(\mathcal{L})}$ -invariant under

$$X^{(\mathcal{L})} = q_1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2}$$

with $\Lambda^{(\mathcal{L})} = \text{diag}(1, 1)$. With

$$w = q_1 q_2, \quad \dot{w} = \dot{q}_1 q_2 + q_1 \dot{q}_2, \quad \zeta = \frac{\dot{q}_1}{q_1} - \log q_1$$

the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \zeta^2 + \frac{1}{2} \frac{\dot{w}^2}{w^2}$$

and the Euler-Lagrange equation for ζ is

$$\partial \mathcal{L} / \partial \zeta = \zeta = 0 \quad \text{or} \quad \dot{q}_1 = q_1 \log q_1$$

with the particular solution

$$q_1(t) = \exp(c e^t) .$$

The corresponding Hamiltonian equations of motion are Λ -symmetric under

$$X = q_1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2} - p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}$$

with $\Lambda = \text{diag}(1, 1, 1, 1)$. Invariants under this X are

$$w_1 = q_1 q_2, \quad w_2 = q_1 p_1, \quad w_3 = q_2 p_2$$

and X is generated by $G = w_2 - w_3$. A “complete” reduction is obtained: with $z = \log q_1$, we get

$$\begin{aligned} \dot{w}_1 &= w_1 w_3 & \dot{w}_2 &= w_3 - w_2 \\ \dot{G} &= -G & \dot{z} &= z + w_2 - w_3 \end{aligned}$$

The above “partial” (Lagrangian) solution $\zeta = 0$ corresponds to

$$\dot{z} = z, \quad w_2 = w_3 = c = \text{const}, \quad \dot{w}_1 = c w_1 .$$

From the Hamiltonian equations, instead, e.g.:

$$q_1(t) = \exp(c e^t) + c_1 \exp(-t) \quad \text{etc.}$$

The reader can easily complete the calculations.

3.4 When $\Lambda^{(\mathcal{L})}$ depends on \dot{q}

If Λ depends also on \dot{q} (see eq.s (4,5)), the calculations performed in subsect. 3.2 cannot be repeated, the coefficient functions ψ_α cannot be expressed in the simple form (6) and the vector field X does not admit a generating function G . In this case one can resort to the other quantity S , introduced in (3), which provides a Λ -constant of motion. An example can completely illustrate this situation.

Example 3 ($n = 1$)

$$\mathcal{L} = \frac{1}{2} \left(\frac{\dot{q}}{q} + 1 \right)^2 \exp(-2q)$$

is $\Lambda^{(\mathcal{L})}$ -invariant under

$$X^{(\mathcal{L})} = q \frac{\partial}{\partial q} \quad \text{with} \quad \Lambda^{(\mathcal{L})} = q + \dot{q} .$$

One finds $\psi = -qp - p$ and the resulting vector field

$$X = q \frac{\partial}{\partial q} - (qp + p) \frac{\partial}{\partial p}$$

does not admit a generating function. Nevertheless, the Hamiltonian equations of motion are Λ -symmetric under X with

$$\Lambda = \begin{pmatrix} q + \dot{q} & 0 \\ -p & q + \dot{q} \end{pmatrix} .$$

Here

$$S = -q$$

satisfies $D_t S = -\nabla(\Lambda\Phi)$ and is a Λ -constant of motion.

4 A digression: general Λ -invariant Lagrangians

The Λ -invariance of a Lagrangian $\mathcal{L} = \mathcal{L}(q, \dot{q}, t)$ considered in subsect. 3.1 is a special case of a much more general situation. Instead of n time-dependent quantities $q_\alpha(t)$, let me consider now n “fields”

$$u_\alpha(x_i) \quad (\alpha = 1, \dots, n; i = 1, \dots, s)$$

depending on $s > 1$ real variables x_i . Now, the Euler-Lagrange equations become a system of PDE's, and the notion of μ -symmetry [12, 13] extends and replaces that of λ -symmetry (or Λ -symmetry if $n > 1$).

In this case, there are $s > 1$ matrices Λ_i ($n \times n$), which must satisfy the compatibility condition

$$D_i \Lambda_j - D_j \Lambda_i + [\Lambda_i, \Lambda_j] = 0 \quad (D_i \equiv D_{x_i}) \quad (7)$$

which can be rewritten putting $\widehat{D}_i = D_i \delta + \Lambda_i$ (or, in explicit form: $(\widehat{D}_i)_{\alpha\beta} = D_i \delta_{\alpha\beta} + (\Lambda_i)_{\alpha\beta}$, with a notation extending the one introduced in Theorem 3),

$$[\widehat{D}_i, \widehat{D}_j] = 0 .$$

Then one has [11, 12, 13]:

Theorem 5 *Given $s > 1$ matrices Λ_i satisfying (7), there exists (locally) a $(n \times n)$ nonsingular matrix Γ such that*

$$\Lambda_i = \Gamma^{-1} (D_i \Gamma) .$$

If a Lagrangian \mathcal{L} is Λ -invariant under a vector field

$$X = \varphi_\alpha \frac{\partial}{\partial u_\alpha}$$

then there is a matrix-valued vector

$$\mathcal{P}_i \equiv (\mathcal{P}_i)_{\alpha\beta}$$

which is Λ -conserved; this Λ -conservation law holds in the form

$$\text{Tr} [\Gamma^{-1} D_i (\Gamma \mathcal{P}_i)] = 0$$

or in the equivalent forms

$$D_i \mathbf{P}_i = -(\Lambda_i)_{\alpha\beta} (\mathcal{P}_i)_{\beta\alpha} = -\text{Tr}(\Lambda_i \mathcal{P}_i) , \quad \text{where} \quad \mathbf{P}_i = (\mathcal{P}_i)_{\alpha\alpha} = \text{Tr} \mathcal{P}_i ,$$

$$\text{Tr}(\widehat{D}_i \mathcal{P}_i) = 0 .$$

For first-order Lagrangians the Λ -conserved “current density vector” \mathcal{P}_i is given by

$$(\mathcal{P}_i)_{\alpha\beta} = \varphi_\alpha \frac{\partial \mathcal{L}}{\partial u_{\beta,i}} \quad \text{where} \quad u_{\beta,i} = \frac{\partial u_\beta}{\partial x_i}$$

and for second-order Lagrangians by

$$(\mathcal{P}_i)_{\alpha\beta} = \varphi_\beta \frac{\partial \mathcal{L}}{\partial u_{\alpha,i}} + ((\widehat{D}_j)_{\beta\gamma} \varphi_\gamma) \frac{\partial \mathcal{L}}{\partial u_{\alpha,ij}} - \varphi_\beta D_j \frac{\partial \mathcal{L}}{\partial u_{\alpha,ij}} .$$

Example 4 *Let $n = s = 2$. Writing for ease of notation, x, y instead of x_1, x_2 , and $u = u(x, y)$, $v = v(x, y)$ instead of u_1, u_2 , consider the vector field*

$$X = u \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \tag{8}$$

and the two matrices

$$\Lambda_1 = \begin{pmatrix} 0 & 0 \\ u_x & 0 \end{pmatrix} \quad \Lambda_2 = \begin{pmatrix} 0 & 0 \\ u_y & 0 \end{pmatrix}$$

and then

$$\Gamma = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} .$$

It is easy to check that the Lagrangian

$$\mathcal{L} = \frac{1}{2}(u_x^2 + u_y^2) - \frac{1}{u}(u_x v_x + u_y v_y) + u^2 \exp(-2v)$$

is Λ -invariant (or better, in this context, μ -invariant) but not invariant under the above vector field X . The μ -conservation law $\text{Tr}(\hat{D}_i \mathcal{P}_i) = 0$ takes here the form

$$D_i \mathbf{P}^i \equiv D_x(uu_x - v_x - \frac{u_x}{u}) + D_y(uu_y - v_y - \frac{u_y}{u}) = u_x^2 + u_y^2 .$$

In agreement with Theorem 5, the r.h.s. of this expression is precisely equal to

$$-\text{Tr}(\Lambda_i \mathcal{P}_i) = -(\Lambda_i \varphi)_\alpha \frac{\partial \mathcal{L}}{\partial u_{\alpha,i}} .$$

Notice that in this case the quantity $u_x^2 + u_y^2$ is just the “symmetry-breaking term”, i.e. the term which prevents the above Lagrangian from being exactly symmetric under the vector field (8).

It should be remarked that μ -symmetries are actually strictly related to *standard* symmetries, or – more precisely – are *locally gauge-equivalent* to them (see for details [11, 4, 14]).

Given indeed the vector field $X = \varphi_\alpha \partial / \partial u_\alpha$ and the s matrices Λ_i , let me denote by

$$X_\Lambda^{(\infty)} = \sum_J \Psi_\alpha^{(J)} \frac{\partial}{\partial u_{\alpha,J}}$$

the infinite Λ -prolongation of X , where the sum is over all multi-indices J as usual, and $\Psi_\alpha^{(0)} = \varphi_\alpha$. Introducing now the other vector field \tilde{X}

$$\tilde{X} \equiv \tilde{\varphi}_\alpha \frac{\partial}{\partial u_\alpha} \quad \text{with} \quad \tilde{\varphi}_\alpha \equiv (\Gamma \varphi)_\alpha$$

where Γ is assigned in Theorem 5, and denoting by

$$\tilde{X}^{(\infty)} = \sum_J \tilde{\varphi}_\alpha^{(J)} \frac{\partial}{\partial u_{\alpha,J}}$$

the *standard* prolongation of \tilde{X} , one has [13, 11] that the coefficient functions $\Psi_\alpha^{(J)}$ of the Λ prolongation of X are connected to the coefficient functions $\tilde{\varphi}_\alpha^{(J)}$ of the standard prolongation of \tilde{X} by the relation

$$\Psi_\alpha^{(J)} = \Gamma^{-1} \tilde{\varphi}_\alpha^{(J)} .$$

In the particularly simple case $n = 1$ (i.e., a single “field” $u(x_i)$), then the $s > 1$ matrices Λ_i , and the matrix Γ as well, become (scalar) functions λ_i and γ ; in this case, if a Lagrangian is μ -invariant under the vector field X , then it is also invariant under the *standard* symmetry $\tilde{X} = \gamma X$. In addition, the μ conservation law can be also expressed as a standard conservation rule

$$D_i \tilde{\mathbf{P}}^i = 0$$

where $\tilde{\mathbf{P}}^i = \gamma \varphi_\alpha \partial \mathcal{L} / \partial u_{\alpha,i}$ is the “current density vector” determined by the vector field $\tilde{X} = \gamma X$.

Example 5 Let now $n = 1$, $s = 2$, and let me introduce for convenience as independent variables the polar coordinates r, θ . I am considering a single “field” $u = u(r, \theta)$ and the rotation vector field $X = \partial / \partial \theta$. The Lagrangian

$$\mathcal{L} = \frac{1}{2} r^2 \exp(-\epsilon \theta) u_r^2 + \frac{1}{2} \exp(\epsilon \theta) u_\theta^2$$

is clearly not invariant under rotation symmetry (if $\epsilon \neq 0$), but is μ -invariant with $\lambda_1 = 0$, $\lambda_2 = \epsilon$. The above Lagrangian is the Lagrangian of a perturbed Laplace equation, indeed the Euler-Lagrange equation is the PDE

$$r^2 u_{rr} + 2r u_r + \exp(2\epsilon \theta) (u_{\theta\theta} + \epsilon u_\theta) = 0 .$$

It is easy to check that the current density vector

$$\mathbf{P} \equiv \left(-r^2 \exp(-\epsilon \theta) u_r u_\theta, \frac{1}{2} r^2 \exp(-\epsilon \theta) u_r^2 - \frac{1}{2} \exp(\epsilon \theta) u_\theta^2 \right)$$

satisfies the μ -conservation law

$$D_i \mathbf{P}_i = -\epsilon \mathbf{P}_2 .$$

According to the above remark on the (local) equivalence of the μ -symmetry X to the standard symmetry $\tilde{X} = \gamma X = \exp(\epsilon \theta) \partial / \partial \theta$, also the (standard) conservation law $D_i \tilde{\mathbf{P}}^i = 0$ holds, with

$$\tilde{\mathbf{P}} \equiv \left(-r^2 u_r u_\theta, \frac{1}{2} r^2 u_r^2 - \frac{1}{2} \exp(2\epsilon \theta) u_\theta^2 \right) .$$

5 Conclusions

I have shown that the notion of λ -symmetry, and the related procedures for studying differential equations, can be conveniently extended to the case of dynamical systems.

The use and the interpretation of this notion becomes particularly relevant when the DS is a Hamiltonian system, and even more if the symmetry is inherited by an invariant Lagrangian: in this context indeed it is possible to introduce in a natural way and to draw a comparison between the notions of Λ -constant of

motion and of Noether Λ -conservation rule. Similarly, the symmetry properties of Euler-Lagrange equations and of the Hamiltonian ones can be compared, and some reduction techniques for the equations can be conveniently introduced.

Finally, I have shown that the Λ -invariance of the Lagrangians in the context of the DS is a special case of a more general and richer situation, where several independent variables are present and a Λ -conservation rule of very general form is true.

Another interesting problem is the nontrivial relationship between λ (or Λ , or μ) symmetries with the standard ones. An aspect of this problem has been mentioned in the above section of this paper. In different situations, this may involve the introduction of nonlocal symmetries and other concepts in differential geometry, as briefly indicated in the Introduction, which clearly go beyond the scope of the present contribution.

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